Re-evaluation of the first formula given in Krueger(1912) for the Gauss-Krueger projection

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For the Gauss-Krueger projection, formulas with power series of longitude difference from the central meridian (e.g. http://www.epsg.org/guides/docs/G7-2.pdf p.36) are normally used. Other kinds of formulas are not found in literatures nowadays. Krueger (1912) is often cited as the original paper of the Gauss-Krueger projection. He gives formulas applicable to very wide zone as the first formula of the Gauss-Krueger projection in the paper. The author evaluates this formula very useful because it can be applied to areas with long distance (thousands of kilometers) from the central meridian including high latitude areas with high accuracy. This will widen the applicability of Gauss-Krueger projection in various fields of geosciences. For example, one may want to handle geospatial data in high latitude area that are distributed in a certain distance but with large longitude differences in one zone of transverse Mercator. Or one may wish to show satellite imagery in one common x-y coordinate system. This formula is also very useful because it is fast and easy to implement in computers. It can be calculated by inputting formulas into cells of a spreadsheet program such as MS Excel.

The formulas are given below with the explanations of variables. Due to the limitations in representing mathematical formulas, the formulas are given as worksheet functions of MS Excel so that one can copy and paste the following formula (right side from the equal sign) and ready to use on the condition that the variables in the expression must be replaced with actual cell addresses that possess numerical values or functions.

B: latitude (radian), L: longitude difference from the central meridian (radian)
e: eccentricity of the earth ellipsoid (For GRS80 ellipsoid, 0.08181919104281579)
A: radius of a circle that has the same perimeter with the meridian ellipse (For GRS80 ellipsoid, 6367449.14577105 m)
g1, g2, g3, g4: coefficients of terms of expansion formula
(For GRS80 ellipsoid, g1=8.3773182473E-04, g2=7.6085277888E-07,
g3=1.1976380192E-09, g4=2.4433762425E-12)
b, eta, xi: variables used in the computation. Definitions are given in the following formulas. 'b' is conformal latitude.
x: plane coordinate (northing), y: plane coordinate (easting)

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b=ATAN(TAN(PI()/4+B/2)*((1-e*SIN(B))/(1+e*SIN(B)))^(e/2))*2-PI()/2
eta=ATANH(SIN(L)*COS(b))
xi=ATAN(TAN(b)/COS(L))
x=A*(xi+g1*SIN(2*xi)*COSH(2*eta)+g2*SIN(4*xi)*COSH(4*eta)+g3*SIN(6*xi)*COSH(6*eta)+g4*SIN(8*xi)*COSH(8*eta))
y=A*(eta+g1*COS(2*xi)*SINH(2*eta)+g2*COS(4*xi)*SINH(4*eta)+g3*COS(6*xi)*SINH(6*eta)+g4*COS(8*xi)*SINH(8*eta))
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Inverse formulas are omitted due to the space limitation. In order to estimate the accuracy of the formulas, x-y coordinates are calculated for several latitude and longitude values and then inverted to latitude and longitude using the inverse formula. The results are given in the following. In the table, the original latitudes and longitudes are given in degrees whereas the discrepancies of them are given in radians. From the results, the maximum error was 3.17E-11 that corresponds to 0.2 mm on the ground.

lat.	lon.	x	y	lat.-difference	lon.-difference
35	2.5	3877450.121018	228245.400450	-3.15E-13	3.19E-16
0	45	0.000000	5627271.912621	0	-7.70E-12
10	50	1703850.492534	6278278.562807	-1.06E-11	2.17E-11
30	60	5455136.341494	6210906.900481	-3.17E-11	1.25E-11
50	70	8217157.794652	4469003.751643	1.91E-12	-5.24E-12
70	80	9598209.995915	2242163.101445	5.20E-14	-2.28E-13

The result clearly shows this formula has enough accuracy for very wide area. It should be noted, however, that users should be aware of distortions due to map projection.

Krueger(1912) writes some parts of these formulas are found in Gauss's posthumous work but he left them uncompleted.

Literature:

Kruger, L. 1912. Konforme Abbildung des Erdellipsoids in der Ebene. Veroffentlichung Koniglich Preuszischen geodatischen Institutes, Neue Folge Nr. 52, 172p.